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Properties of a Job Search Problem on a Partially Observable Markov Chain in a Dynamic Economy

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Abstract—This paper observes a job search problem on a partially observable Markov chain, which can be considered as an extension of a job search in a dynamic economy in [1]. This problem is formulated as the state changes according to a partially observable Markov chain, i.e., the current state cannot be observed but there exists some information regarding what a present state is. All information about the unobservable state are summarized by the probability distributions on the state space, and we employ the Bayes' theorem as a learning procedure. The total positivity of order two, or simply TP_2, is a fundamental property to investigate sequential decision problems, and it also plays an important role in the Bayesian learning procedure for a partially observable Markov process. By using this property, we consider some relationships among prior and posterior information, and the optimal policy. We will also observe the probabilities to make a transition into each state after some additional transitions by employing the optimal policy. In the stock market, suppose that the states correspond to the business situation of one company and if there is a state designating the default, then the problem is what time the stocks are sold off before bankrupt, and the probability to become bankrupt will be also observed.

Keywords—Partially observable Markov chain, Total positivity, Job search problem.

1. INTRODUCTION

This paper will observe a job search problem on a partially observable Markov chain and will consider the probability to make a transition into each state after some additional transitions by employing the optimal policy. This is one of the optimal stopping problems and can be considered as an extension of a job search in a dynamic economy discussed by Lippmann and MacCall [1]. For instance, in economics, we consider that the conditions of economy are divided into some classes, and assume them to be getting worse. Let us assume that these conditions are not directly observable. That is, it cannot be known which one of these classes it is now, but there is some information regarding what a present class is. When each state of this Markov chain corresponds the class of the economy, we suppose that the wages of a job are a random variable depending on these classes. Differing from the case in [1], the state changes according to a partially observable Markov chain. On the other hand, in the stock market, we consider these classes to correspond to the business situation of one company, i.e., these conditions can be

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estimated through the stock price of this company in the stock market. Since the stock price is observable, the problem is what time the stocks are sold off. For a job search problem in which the current state is observable, it is known that the maximization is achieved by classifying all possible job offers into two mutually exclusive classes, and the wage of a job offer that separates these two classes is called the reservation wage. It is not, however, always true for this problem since the state of the chain is unobservable for the decision maker.

All information about the unobservable state is summarized by probability distributions on the state space, and we employ the Bayes' theorem as a learning procedure. The total positivity of order two, or simply TP_2, is a fundamental property to investigate sequential decision problems, and it also plays an important role in the Bayesian learning procedure for a partially observable Markov process. By using this property, we consider some relationships among prior and posterior informations, the optimal policy and the probabilities to make a transition into each state. The properties of this TP_2 are also investigated by Karlin and McGregor [2], Karlin [3], Karlin and Rinott [4], and by others regarding the stochastic processes.

In order to observe these probabilities when we employ the Bayes' theorem as a learning procedure, we will start to reconsider a job search in a dynamic economy to compare the properties of this problem on a partially observable Markov chain. In Section 2, we summarize the properties of a job search problem when the state of the chain is directly observable. It will be shown that the probabilities to make a transition into each state after some additional transitions is TP_2. In Section 3, we will investigate the probabilities to make a transition into each state when the state changes according to a partially observable Markov chain. We will also observe the similar probabilities by employing the optimal policy of the job search problem. Suppose that the State i represents the class of the business situation of one company ($i \in \{1, 2, \ldots, K\}$), and we specially suppose that the State K designates the default. Then the problem is what time the stocks are sold off before bankrupt, and the probability to become bankrupt is also observed by these considerations.

2. JOB SEARCH IN A DYNAMIC ECONOMY

2.1. Optimal Policy and the Expected Reward

Consider a finite state Markov chain with the state space $\{1, 2, \ldots, K\}$ with the transition probability $\mathbf{P} = (p_{ij})_{i,j=1,2,\ldots,K}$. The job search is a problem to find a job in order to maximize the expected reward without recall, and the jobs appear one at a time in sequential order. When each State i of this Markov chain corresponds the class of economy, let X_i be a nonnegative real valued random variable representing the wage of a job associated to this condition ($i = 1, 2, \ldots, K$). When the decision-maker knows what a present state is, Lippmann and MacCall [1] considered this problem in a dynamic economy under two conditions (1) and (2):

(1) X_i is stochastically increasing in i, i.e., $F_1(x) \geq F_2(x) \geq \cdots \geq F_K(x)$ for all x,
(2) $\sum_{j=k}^{K} p_{ij}$ is increasing in i for all k ($k = 1, 2, \ldots, K$).

In this paper, we treat this job search problem under an uncertainty condition, i.e., the state of this chain cannot be observed directly. Since the TP_2 plays an important role in the Bayesian learning procedure, we introduce two assumptions concerning the transition probability and the distribution function of X_i as Assumptions 1 and 2 ($i = 1, 2, \ldots, K$), which are the differences to the Lippmann and MacCall's case. These nonnegative random variables X_i are absolutely continuous with the density function $f_i(x)$ ($i = 1, 2, \ldots, K$). It is possible to generalize this problem for a partially observable Markov process as is observed by Nakai [5], and also apply it to the sequential decision problems (see [6–8] and so on). In Definition 1, we introduce a stochastic relation among random variables defined on a complete separable metric space with total order $\geq$.

DEFINITION 1. *Suppose that two random variables X and Y have the respective probability density functions $f(x)$ and $g(x)$. If $f(y)g(x) \leq f(x)g(y)$ for all x and y where $x \geq y$, then X is said to be greater than Y by means of the likelihood ratio, or simply $X \succeq Y$.*

DEFINITION 2. *Suppose that there is a $K \times K$ matrix $\mathbf{A} = (a_{ij})_{i,j=1,2,\ldots,K}$. If*

$$\begin{vmatrix} a_{ik} & a_{il} \\ a_{jk} & a_{jl} \end{vmatrix} \geq 0$$

for all i, j, k and l where $i \leq j$ and $k \leq l$ $(i, j, k, l = 1, 2, \ldots, K)$, then this matrix $\mathbf{A}$ is said to be total positive of order two, or simply TP_2.

It is easy to show that the order defined by Definition 1 is a partial order, and it is also said to be TP_2. In order to investigate the problem treated here, we introduce the following two assumptions. Since we employ the Bayes' theorem as the learning procedure, these assumptions are necessary to analyze this problem.

ASSUMPTION 1. *If $k \leq l$, then $X_k \succeq X_l$ $(k, l = 1, 2, \ldots, K)$, i.e., X_i is decreasing with respect to i by means of the likelihood ratio.*

ASSUMPTION 2. *The transition probability matrix $\mathbf{P} = (p_{ij})_{i,j=1,2,\ldots,K}$ is* TP_2.

In Assumption 1, $X_k \succeq X_l$ implies that if $x > y$, then $f_k(y)f_l(x) \leq f_k(x)f_l(y)$ for all k and l where $k \leq l$ $(k, l = 1, 2, \ldots, K)$, and this implies that X_i is stochastically decreasing in i. And, therefore, the random variable X_i takes on smaller values as i becomes larger, and an example of this is where State 1 represents the highest class, State 2 the second highest, ..., and State K is the lowest class. Assumption 2 is known as TP_2 for this Markov chain. This implies that the probability of moving from the current state to 'better' states decreases with improvement in the current state. By this assumption, as the number i associated with each state becomes larger, the probability to make a transition into the lower class increases. For the state space $\{1, 2, \ldots, K\}$, each state represents the class of the business situation of one company, and suppose that $p_{KK} = 1$ for the transition probability $\mathbf{P}$ and $X_K \equiv 0$ with probability 1, then it is possible to consider that State K designates a default.

When n jobs remain (i.e., there are n stages to go) and a wage of the currently available job offer is x, if this job is accepted, then a reward $u_n(x)$ will be obtained. We say we are in State (i, x) if the economy is in State i and a wage of the currently available job offer is x. The cost c is necessary to search a next job offer, and we introduce a discount factor $0 < \beta < 1$. Let $v_n(i, x)$ be a maximal β-discount expected reward earned by the optimal policy when n jobs remain and we are in State (i, x), then the principle of optimality implies the following recursive equation of $v_n(i, x)$.

$$v_n(i,x) = \max\left\{ u_n(x), -c + \beta \sum_{j=1}^{K} p_{ij} \int_0^\infty v_{n-1}(j,y)\, dF_j(y) \right\} \tag{1}$$

with $v_1(i, x) = u_1(x)$. We assume $u_n(x)$ to be an increasing function of x and n. For example, $u_n(x) = ((1 - \delta^n)/(1 - \delta))x$ satisfies these conditions, and this function is an amount of the total sum of the capital and interest when we make a deposit of x at an annual interest rate of γ for n years where $\delta = 1 + \gamma$. It is easy to show that the maximization is achieved by classifying all possible job offers into two mutually exclusive classes. The wage of a job offer that separates these two classes is called the reservation wage, and we denote this reservation wage as $\alpha_n(i)$ when n jobs remain and the current state of the chain is i. Under Assumptions 1 and 2 and the assumption regarding $u_n(x)$, these reservation wages $\alpha_n(i)$ and the maximal β-discount expected reward $v_n(i, x)$ satisfy the following properties by the induction principle.

LEMMA 1. *For any state $i \in \{1, 2, \ldots, K - 1\}$ and positive integer n, $\alpha_n(i) \geq \alpha_n(i + 1)$, and $\alpha_{n+1}(i) \geq \alpha_n(i)$ for all $i \in \{1, 2, \ldots, K\}$ and positive integer n.*

LEMMA 2. *For any state* $i \in \{1,2,\dots,K\}$ *and positive integer* n, $v_{n+1}(i,x) \ge v_n(i,x)$ *and* $v_{n+1}(i,x) \ge v_{n+1}(i+1,x)$ $(x>0)$. *If* $x>y$, *then* $v_{n+1}(i,x) \ge v_{n+1}(i,y)$.

2.2. Probability to Make a Transition into Each State

In this section, we consider a probability to make a transition into State j after n additional transitions when the state of the chain can be observable in order to compare the properties of the job search problem on a partially observable Markov chain with this case. First, we only consider a change of the states of this chain. When the state of the chain is in State i, let $p^*_{ij}(n)$ be the probability to make a transition into State j after n additional transitions ($i,j=1,2,\dots,K$ and $n=1,2,\dots$). It is easy to show that the probability $p^*_{ij}(n)$ satisfies the recursive equation $p^*_{ij}(n)=\sum_{k=1}^{K} p_{ik}p^*_{kj}(n-1)$ with the initial condition $p^*_{ij}(1)=p_{ij}$. If we put a $K\times K$ matrix as $\mathbf{P}^*(n)=(p^*_{ij}(n))$, then this recursive equation can be represented as $\mathbf{P}^*(n)=\mathbf{P}\mathbf{P}^*(n-1)$ with $\mathbf{P}^*(1)=\mathbf{P}$.

LEMMA 3. *If the matrices* $\mathbf{A}$ *and* $\mathbf{B}$ *are* TP_2, *then the product* $\mathbf{AB}$ *is also* TP_2.

If $\mathbf{P}^*(n-1)$ is shown to be TP_2, then the induction principle on n and Lemma 3 imply that $\mathbf{P}^*(n)=(p^*_{ij}(n))$ is TP_2 since $\mathbf{P}=(p_{ij})$ is TP_2 by Assumption 2.

We will next observe the similar probabilities when the decision-maker employs the optimal policy. When the current state is i and n jobs remain (i.e., there are n stages to go), let $\tilde{p}_{ij}(n,m)$ be the probability to make a transition into State j after m additional transitions by employing the optimal policy ($i,j=1,2,\dots,K$ and $m\le n$, $m,n=1,2,\dots$). For this job search problem, since the optimal policy is determined by the reservation wages $\alpha(i,n)$, $F_i(\alpha(i,n))$ is a probability not to accept the current job offer when the state is i and n jobs remain. It is, therefore, easy to show that $\tilde{p}_{ij}(n,m)$ satisfies the recursive equation

$$\tilde{p}_{ij}(n,m)=F_i(\alpha(i,n))\sum_{k=1}^{K} p_{ik}\tilde{p}_{kj}(n-1,m-1), \tag{2}$$

with the initial condition $\tilde{p}_{ij}(n,1)=F_i(\alpha(i,n))p_{ij}$. If we put $\tilde{\mathbf{P}}(n,m)=(\tilde{p}_{ij}(n,m))$, then $\tilde{\mathbf{P}}(n,1)=\mathbf{D}(F_1(\alpha(1,n)),\dots,F_K(\alpha(K,n)))\mathbf{P}$ for all n (>0), and equation (2) is equal to

$$\tilde{\mathbf{P}}(n,m)=\mathbf{D}(F_1(\alpha(1,n)),\dots,F_K(\alpha(K,n)))\mathbf{P}\tilde{\mathbf{P}}(n-1,m-1), \tag{3}$$

where $\mathbf{D}(F_1(\alpha(1,n)),\dots,F_K(\alpha(K,n)))$ is a diagonal matrix

$$\mathbf{D}(F_1(\alpha(1,n)),\dots,F_K(\alpha(K,n)))=\begin{pmatrix} F_1(\alpha(1,n)) & 0 & \cdots & 0 \\ 0 & F_2(\alpha(2,n)) & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & F_K(\alpha(K,n)) \end{pmatrix}.$$

As a corollary of Lemma 3, the following property is obtained, and this corollary implies Proposition 1 concerning $\tilde{\mathbf{P}}(n,m)$.

COROLLARY 1. *If a matrix* $\mathbf{A}$ *is* TP_2 *and a* $\mathbf{D}$ *is a diagonal matrix, then* $\mathbf{DA}$ *is also* TP_2.

PROPOSITION 1. $\tilde{\mathbf{P}}(n,m)=(\tilde{p}_{ij}(n,m))$ *is* TP_2.

PROOF. We employ the induction principle on m. When $m=1$, Corollary 1 implies that $\tilde{\mathbf{P}}(n,1)$ is TP_2 since $\tilde{\mathbf{P}}(n,1)=\mathbf{D}(F_1(\alpha(1,n)),\dots,F_K(\alpha(K,n)))\mathbf{P}$. Assume that $\tilde{\mathbf{P}}(n,m)$ is TP_2 for all value less than m. Since $\mathbf{P}=(p_{ij})$ and $\tilde{\mathbf{P}}(n-1,m-1)$ are TP_2, equation (2) and Corollary 1 imply that $\tilde{\mathbf{P}}(n,m)=(\tilde{p}_{ij}(n,m))$ is also TP_2. ∎

3. JOB SEARCH WITH INCOMPLETE INFORMATION

3.1. Optimal Policy and the Expected Reward

Differing from the previous section, we will deal with a job search where the state cannot be observed directly, i.e., we will consider this problem on a partially observable Markov chain. Information about the unobservable state is assumed to be a probability distribution $\boldsymbol{\mu}$ on the state space $\{1, 2, \ldots, K\}$. Let $\mathcal{S}$ be a set of all information about the unobservable state of this chain, then

$$\mathcal{S} = \left\{ \boldsymbol{\mu} = (\mu_1, \mu_2, \ldots, \mu_K) \;\middle|\; \sum_{i=1}^{K} \mu_i = 1,\ \mu_s \geq 0 \ (s = 1, 2, \ldots, K) \right\}.$$

Among informations in $\mathcal{S}$, we define an order by Definition 3. It is possible to generalize this order relation in order to investigate a partially observable Markov process, and the details regarding this process are shown in [5,9] with the applications to the sequential decision problems.

DEFINITION 3. *For the probability distributions $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$ on the set $\{1, 2, \ldots, K\}$, if $\mu_j \nu_i \geq \mu_i \nu_j$ for all i and j where $i \leq j$ $(i, j = 1, 2, \ldots, K)$ and if $\mu_j \nu_i > \mu_i \nu_j$ at least one pair of i and j, then $\boldsymbol{\mu}$ is said to be greater than $\boldsymbol{\nu}$ by means of the likelihood ratio, or simply $\boldsymbol{\mu} \succ \boldsymbol{\nu}$.*

By Definition 3, if $\boldsymbol{\mu} \succeq \boldsymbol{\nu}$, then, as j becomes larger, the ratio μ_j / ν_j of the probabilities taking on the value j increases. On the other hand, since $\mathbf{P}$ satisfies Assumption 2, if we put $\mathbf{p}_i = (p_{i\,1}, \ldots, p_{i\,K})$ $(i = 1, 2, \ldots, K)$, then $\mathbf{p}_{i+1} \succeq \mathbf{p}_i$ for all $i \in \{1, 2, \ldots, K-1\}$.

Regarding the unobservable state of the chain, it is assumed that there exists an information system or an observation process to obtain information about it. Since the random variables $\{X_i \mid i = 1, 2, \ldots, S\}$ indicate the wage of a job offer depending on the unobservable state of the chain, it can be considered as an information system of this process, i.e., we improve information about the unobservable state by using a wage of a current job offer. When prior information is $\boldsymbol{\mu}$, we first observe a wage of a current job offer depending on the state and improve information about it by using the Bayes' theorem as the learning procedure. After that, we see time moving forward by one unit and thus this chain will make a transition into a new state. It is also possible to formulate and analyze this model by other order. If the wage of a current job offer is x, we improve information as $\boldsymbol{\mu}(x) = (\mu(x)_1, \mu(x)_2, \ldots, \mu(x)_K)$ by employing the Byes' theorem, and, after changing to a new state according to $\mathbf{P}$, prior information at the next stage will be $\overline{\boldsymbol{\mu}(x)}$.

For a vector valued function $\mathbf{h}(x)$, a monotonic property is introduced as Definition 4.

DEFINITION 4. *For a vector valued nonnegative function $\mathbf{h}(x) = (h_1(x), h_2(x), \ldots, h_K(x))$ of x, if $x > y$ then $\mathbf{h}(y) \succeq \mathbf{h}(x)$ (or $\mathbf{h}(x) \succeq \mathbf{h}(y)$), i.e., $h_j(y)h_i(x) \geq h_i(y)h_j(x)$ (or $h_j(y)h_i(x) \leq h_i(y)h_j(x)$) for all i and j where $i \leq j$ $(i, j = 1, 2, \ldots, K)$), then this function $\mathbf{h}(x)$ is said to be a decreasing (or an increasing) function of x.*

Since the random variables $\{X_i \mid i = 1, 2, \ldots, K\}$ with the density functions $\{f_i(x) \mid i = 1, 2, \ldots, K\}$ satisfy Assumption 1, $\mathbf{f}(y) \succeq \mathbf{f}(x)$ where $\mathbf{f}(x) = (f_1(x), \ldots, f_K(x))$, and, therefore, $\mathbf{f}(x)$ is an increasing function of x.

Regarding the relationship between prior information $\boldsymbol{\mu}$ and posterior information $\overline{\boldsymbol{\mu}(x)}$, the following essential properties can be obtained under Assumptions 1 and 2, which is known as Lemma 4 of Nakai [5] and so on.

LEMMA 4. *If $\boldsymbol{\mu} \succ \boldsymbol{\nu}$, then $\boldsymbol{\mu}(x) \succ \boldsymbol{\nu}(x)$ and $\overline{\boldsymbol{\mu}(x)} \succ \overline{\boldsymbol{\nu}(x)}$ for all x. For any $\boldsymbol{\mu}$, then $\boldsymbol{\mu}(x)$ and $\overline{\boldsymbol{\mu}(x)}$ is a decreasing function of x.*

Lemma 4 implies that the order relation (Definition 3) among prior information $\boldsymbol{\mu}$ is preserved in improved information $\boldsymbol{\mu}(x)$ and posterior information $\overline{\boldsymbol{\mu}(x)}$. Furthermore, for the same prior information $\boldsymbol{\mu}$, as a wage x of a job offer increases, posterior information $\overline{\boldsymbol{\mu}(x)}$ becomes worse by means of Definition 3.

Suppose a job search problem with prior information $\boldsymbol{\mu}$ about the unobservable state, and let $v_n(\boldsymbol{\mu}, x)$ be a maximal β-discount expected reward attainable when there are n stages to go and the currently available job offer is x $(0 < \beta < 1)$. The principle of optimality yields the recursive equation of $v_n(\boldsymbol{\mu}, x)$ as

$$v_n(\boldsymbol{\mu}, x) = \max\left\{u_n(x), c + \beta \int_0^\infty v_{n-1}\left(\overline{\boldsymbol{\mu}(x)}, y\right) dF_{\overline{\boldsymbol{\mu}(x)}}(y)\right\}, \tag{4}$$

with $v_1(\boldsymbol{\mu}, x) = \mathrm{E}_{\boldsymbol{\mu}}[u_1(X)] = \int_0^\infty u_1(x)\, dF_{\boldsymbol{\mu}}(x)$ in which $F_{\boldsymbol{\mu}}(x) = \sum_{i=1}^K \mu_i F_i(x)$ is known as a weighted distribution function [10].

Put $S(\boldsymbol{\mu}, n) = \{x \mid u_n(x) \geq c + \beta \int_0^\infty v_{n-1}(\overline{\boldsymbol{\mu}(x)}, y)\, dF_{\overline{\boldsymbol{\mu}(x)}}(y)\}$ and $C(\boldsymbol{\mu}, n) = S(\boldsymbol{\mu}, n)^c$, then these sets $S(\boldsymbol{\mu}, n)$ and $C(\boldsymbol{\mu}, n)$ correspond to a stopping and continuance region, respectively, for this job search problem. Lemma 5 is derived from the fact that $\overline{\boldsymbol{\mu}(x)}$ decreases as x increases.

LEMMA 5. *For all $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$ where $\boldsymbol{\mu} \succeq \boldsymbol{\nu}$, if $h(x)$ is an increasing function of x, then $\int_0^\infty h(x)\, dF_{\boldsymbol{\mu}} \cdot (x) \leq \int_0^\infty h(x)\, dF_{\boldsymbol{\nu}}(x)$.*

If the function $v_{n-1}(\boldsymbol{\mu}, z)$ is an increasing function of z and a decreasing function of $\boldsymbol{\mu}$, then Lemmas 4 and 5 imply

$$\int_0^\infty v_{n-1}\left(\overline{\boldsymbol{\mu}(x)}, z\right) dF_{\overline{\boldsymbol{\mu}(x)}}(z) \geq \int_0^\infty v_{n-1}\left(\overline{\boldsymbol{\mu}(y)}, z\right) dF_{\overline{\boldsymbol{\mu}(y)}}(z) \tag{5}$$

for all x and y where $x > y$. Concerning two regions $S(\boldsymbol{\mu}, n)$ and $C(\boldsymbol{\mu}, n)$, equation (5) implies the following property.

LEMMA 6. *If $\boldsymbol{\mu} \succeq \boldsymbol{\nu}$ where $\boldsymbol{\mu}, \boldsymbol{\nu} \in \mathcal{S}$, then $S(\boldsymbol{\nu}, n) \subset S(\boldsymbol{\mu}, n)$ and $S(\boldsymbol{\mu}, n+1) \subset S(\boldsymbol{\mu}, n)$.*

Since $S(\boldsymbol{\mu}, n) \cup C(\boldsymbol{\mu}, n) = \Re_+ = [0, \infty)$ and $S(\boldsymbol{\mu}, n) \cap C(\boldsymbol{\mu}, n) = \emptyset$ for all $\boldsymbol{\mu}$ and $n \geq 1$, this lemma implies $C(\boldsymbol{\mu}, n) \subset C(\boldsymbol{\nu}, n)$ and $C(\boldsymbol{\mu}, n) \subset C(\boldsymbol{\mu}, n+1)$. It can be shown that $v_n(\boldsymbol{\mu}, x)$ satisfies Lemma 7 by the induction principle on n as Nakai [5] and so on.

LEMMA 7. *If $\boldsymbol{\mu} \succeq \boldsymbol{\nu}$ where $\boldsymbol{\mu}, \boldsymbol{\nu} \in \mathcal{S}$, then $v_n(\boldsymbol{\mu}, x) \leq v_n(\boldsymbol{\nu}, x)$. If $x > y$ where $x, y \in \Re_+$, then $v_{n+1}(\boldsymbol{\mu}, x) \geq v_n(\boldsymbol{\mu}, x)$ and $v_n(\boldsymbol{\mu}, x) \geq v_n(\boldsymbol{\mu}, y)$.*

3.2. Probability to Make a Transition into Each State

Similar to Section 2.2, we will consider a probability to make a transition into State j after n additional transitions when the state changes according to a partially observable Markov chain. Initially, we observe these probabilities leaving the decision and the learning procedure regarding the unobservable state in order to consider these probabilities in sequential order. When prior information about the unobservable state is $\boldsymbol{\mu}$, let $\mathbf{P}^*(\boldsymbol{\mu}, m) = (P^*(\boldsymbol{\mu}, m)_1, \ldots, P^*(\boldsymbol{\mu}, m)_K)$ be a probability vector to make a transition into State j after m additional transitions. When $m = 1$, $\mathbf{P}^*(\boldsymbol{\mu}, 1) = (\sum_{i=1}^K \mu_i p_{i1}, \ldots, \sum_{i=1}^K \mu_i p_{iK}) = \boldsymbol{\mu}\mathbf{P} = \bar{\boldsymbol{\mu}}$, and $\mathbf{P}^*(\boldsymbol{\mu}, 2) = \mathbf{P}^*(\bar{\boldsymbol{\mu}}, 1) = \bar{\boldsymbol{\mu}}\mathbf{P} = \boldsymbol{\mu}\mathbf{P}^2$ for $m = 2$. We, therefore, obtain

$$\mathbf{P}^*(\boldsymbol{\mu}, m) = \mathbf{P}^*(\bar{\boldsymbol{\mu}}, m-1) = \mathbf{P}^*(\boldsymbol{\mu}\mathbf{P}, m-1) = \boldsymbol{\mu}\mathbf{P}^m. \tag{6}$$

Since $\mathbf{P}$ is TP_2, Lemma 3 and the induction principle on m implies that $\mathbf{P}^*(\boldsymbol{\mu}, m) = \boldsymbol{\mu}\mathbf{P}^m$ is also TP_2. By using Lemma 8, equation (6) yields Proposition 2.

LEMMA 8. *If $\boldsymbol{\mu} \succeq \boldsymbol{\nu}$ where $\boldsymbol{\mu}, \boldsymbol{\nu} \in \mathcal{S}$ and $\mathbf{A} = (a_{ij})$ is TP_2, then $\boldsymbol{\mu}\mathbf{A} \succeq \boldsymbol{\nu}\mathbf{A}$.*

PROPOSITION 2. *If $\boldsymbol{\mu} \succeq \boldsymbol{\nu}$ where $\boldsymbol{\mu}, \boldsymbol{\nu} \in \mathcal{S}$, then $\mathbf{P}^*(\boldsymbol{\mu}, m) \succeq \mathbf{P}^*(\boldsymbol{\nu}, m)$.*

Next, we treat similar probabilities leaving the decision out of consideration, i.e., we take account of the learning procedure regarding the unobservable state by using an observed wage of a currently available job offer. When prior information is $\boldsymbol{\mu}$, we first observe a wage depending on the current state and improve information about it by using the Bayes' theorem. After that, we see time moving forward by one unit and thus this chain will make a transition into a new state according to $\mathbf{P}$. Hence, whenever we say prior information about the state is $\boldsymbol{\mu}$, the transition into the current state has been finished. For this case, let $\hat{P}(\boldsymbol{\mu}, m)_j$ be a probability to make a transition into State j after m additional transitions ($j = 1, 2, \ldots, K$), and $\hat{\mathbf{P}}(\boldsymbol{\mu}, m) = (\hat{P}(\boldsymbol{\mu}, m)_1, \ldots, \hat{P}(\boldsymbol{\mu}, m)_K)$.

For a vector valued function $\mathbf{u}(x) = (u_1(x), \ldots, u_K(x))$ where $\int_a^b u_i(x)\, dF(x)$ exists for all i ($= 1, 2, \ldots, K$), we use a notation $\int_a^b \mathbf{u}(x)\, dF(x)$ as $\int_a^b \mathbf{u}(x)\, dF(x) = (\int_a^b u_1(x)\, dF(x), \ldots, \int_a^b u_K(x)\, dF(x))$ for the sake of simplicity in the subsequent discussions.

When prior information about the unobservable state of the chain is $\boldsymbol{\mu}$, since $\hat{\mathbf{P}}(\boldsymbol{\mu}, 1)$ is a probability distribution on the state space at the next stage, we have

$$\hat{\mathbf{P}}(\boldsymbol{\mu}, 1) = \int_0^\infty \boldsymbol{\mu}(x)\mathbf{P}\, dF_{\boldsymbol{\mu}}(x) = \int_0^\infty \overline{\boldsymbol{\mu}(x)}\, dF_{\boldsymbol{\mu}}(x). \tag{7}$$

If prior information is $\boldsymbol{\mu}$ and a wage x of a current job offer is observed, posterior information will be $\overline{\boldsymbol{\mu}(x)}$ at the next stage. For $m = 2$, $\hat{\mathbf{P}}(\boldsymbol{\mu}, 2) = \int_0^\infty \hat{\mathbf{P}}(\overline{\boldsymbol{\mu}(x)}, 1)\, dF_{\boldsymbol{\mu}}(x)$ where $\hat{\mathbf{P}}(\boldsymbol{\mu}, 2)$ is a probability distribution on the state space after two stages. Similarly to these cases, when prior information is $\boldsymbol{\mu}$, $\hat{\mathbf{P}}(\boldsymbol{\mu}, m)$ satisfies equation (8), since $\hat{\mathbf{P}}(\boldsymbol{\mu}, m)$ is a probability distribution on the state space after m stages.

$$\hat{\mathbf{P}}(\boldsymbol{\mu}, m) = \int_0^\infty \hat{\mathbf{P}}\left(\overline{\boldsymbol{\mu}(x)}, m-1\right)\, dF_{\boldsymbol{\mu}}(x), \tag{8}$$

with equation (7) for $m = 1$. In order to observe these probabilities, we introduce an order in Definition 5.

Definition 5. *For two vector valued nonnegative functions $\mathbf{g}(x) = (g_1(x), \ldots, g_K(x))$ and $\mathbf{h}(x) = (h_1(x), \ldots, h_K(x))$ of x, if $g(x)_j h(x)_i \ge g(x)_i h(x)_j$ for all i and j where $i \le j$ ($i, j = 1, 2, \ldots, K$), then $\mathbf{g}(x)$ is said to be greater than $\mathbf{h}(x)$ by means of* TP_2, *or simply $\mathbf{g}(\mathbf{x}) \succeq \mathbf{h}(\mathbf{x})$.*

Concerning this order, Lemmas 9, 10, and Corollary 2 can be obtained. Since the inequality $\boldsymbol{\mu} \succeq \boldsymbol{\nu}$ implies $\bar{\boldsymbol{\mu}} \succeq \bar{\boldsymbol{\nu}}$ and $\overline{\boldsymbol{\mu}(x)} \succeq \overline{\boldsymbol{\nu}(x)}$ by Lemma 4, Corollaries 3 and 4 can be shown regarding the monotonic property of the integration of the vector valued function. By preparing these corollaries, the property regarding the probability $\hat{\mathbf{P}}(\boldsymbol{\mu}, m)$ can be obtained in Proposition 3.

Lemma 9. *If $\mathbf{g}(x) = (g(x)_1, \ldots, g(x)_K)$ and $\mathbf{h}(x) = (h(x)_1, \ldots, h(x)_K)$ are decreasing functions of x and $\mathbf{g}(x) \succeq \mathbf{h}(x)$, then $\int_0^\infty \mathbf{g}(x)\, dF(x) \succeq \int_0^\infty \mathbf{h}(x)\, dF(x)$.*

Corollary 2. *If $\boldsymbol{\mu} \succeq \boldsymbol{\nu}$ where $\boldsymbol{\mu}, \boldsymbol{\nu} \in \mathcal{S}$, then $\int_0^\infty \overline{\boldsymbol{\mu}(x)}\, dF(x) \succeq \int_0^\infty \overline{\boldsymbol{\nu}(x)}\, dF(x)$.*

Lemma 10. *If $\boldsymbol{\mu} \succeq \boldsymbol{\nu}$ where $\boldsymbol{\mu}, \boldsymbol{\nu} \in \mathcal{S}$ and a vector valued function $\mathbf{h}(x)$ is a decreasing function of x, then $\int_0^\infty \mathbf{h}(x)\, dF_{\boldsymbol{\mu}}(x) \succeq \int_0^\infty \mathbf{h}(x)\, dF_{\boldsymbol{\nu}}(x)$.*

Corollary 3. *If $\boldsymbol{\mu} \succeq \boldsymbol{\nu}$ and $\mathbf{g}(x) \succeq \mathbf{h}(x)$, then $\int_0^\infty \mathbf{g}(x)\, dF_{\boldsymbol{\mu}}(x) \succeq \int_0^\infty \mathbf{h}(x)\, dF_{\boldsymbol{\nu}}(x)$.*

Corollary 4. *If $\boldsymbol{\mu} \succeq \boldsymbol{\nu}$, then $\int_0^\infty \mathbf{h}(\boldsymbol{\mu}, x)\, dF_{\boldsymbol{\mu}}(x) \succeq \int_0^\infty \mathbf{h}(\boldsymbol{\nu}, x)\, dF_{\boldsymbol{\nu}}(x)$ where $\mathbf{h}(\boldsymbol{\mu}, x)$ is an increasing function of $\boldsymbol{\mu}$ and a decreasing function of x.*

Proposition 3. *If $\boldsymbol{\mu} \succeq \boldsymbol{\nu}$, then $\hat{\mathbf{P}}(\boldsymbol{\mu}, m) \succeq \hat{\mathbf{P}}(\boldsymbol{\nu}, m)$, i.e., $\hat{\mathbf{P}}(\boldsymbol{\mu}, m)$ is an increasing function of $\boldsymbol{\mu}$.*

Proof. We employ the induction principle on m. When $m = 1$, if $\boldsymbol{\mu} \succeq \boldsymbol{\nu}$, then $\hat{\mathbf{P}}(\boldsymbol{\mu}, 1) = \int_0^\infty \overline{\boldsymbol{\mu}(x)}\, dF_{\boldsymbol{\mu}}(x)$ and Corollary 2 imply $\hat{\mathbf{P}}(\boldsymbol{\mu}, 1) \succeq \hat{\mathbf{P}}(\boldsymbol{\nu}, 1)$. Since $\overline{\boldsymbol{\mu}(x)} \succeq \overline{\boldsymbol{\nu}(x)}$, we have $\hat{\mathbf{P}}(\overline{\boldsymbol{\mu}(x)}, 1) \succeq \hat{\mathbf{P}}(\overline{\boldsymbol{\nu}(x)}, 1)$, and Corollary 4 implies

$$\hat{\mathbf{P}}(\boldsymbol{\mu}, 2) = \int_0^\infty \hat{\mathbf{P}}\left(\overline{\boldsymbol{\mu}(x)}, 1\right)\, dF_{\boldsymbol{\mu}}(x) \succeq \int_0^\infty \hat{\mathbf{P}}\left(\overline{\boldsymbol{\nu}(x)}, 1\right)\, dF_{\boldsymbol{\mu}}(x) = \hat{\mathbf{P}}(\boldsymbol{\nu}, 2),$$

and, therefore, $\hat{\mathbf{P}}(\boldsymbol{\mu},2) \succeq \hat{\mathbf{P}}(\boldsymbol{\nu},2)$.

By the induction assumption, if $\boldsymbol{\mu} \succeq \boldsymbol{\nu}$, then $\hat{\mathbf{P}}(\boldsymbol{\mu},m-1) \succeq \hat{\mathbf{P}}(\boldsymbol{\nu},m-1)$. Since $\overline{\boldsymbol{\mu}(x)} \succeq \overline{\boldsymbol{\nu}(x)}$, we also have $\hat{\mathbf{P}}(\overline{\boldsymbol{\mu}(x)},m-1) \succeq \hat{\mathbf{P}}(\overline{\boldsymbol{\nu}(x)},m-1)$. Corollary 4 implies

$$\begin{aligned}\hat{\mathbf{P}}(\boldsymbol{\mu},m) &= \int_0^\infty \hat{\mathbf{P}}\left(\overline{\boldsymbol{\mu}(x)},m-1\right) dF_{\boldsymbol{\mu}}(x)\\ &\succeq \int_0^\infty \hat{\mathbf{P}}\left(\overline{\boldsymbol{\nu}(x)},m-1\right) dF_{\boldsymbol{\mu}}(x)\\ &\succeq \int_0^\infty \hat{\mathbf{P}}\left(\overline{\boldsymbol{\nu}(x)},m-1\right) dF_{\boldsymbol{\nu}}(x) = \hat{\mathbf{P}}(\boldsymbol{\nu},m),\end{aligned}$$

and the proof is completed. ∎

Finally, we will consider similar probabilities by giving consideration to the learning procedure and the optimal policy. When prior information is $\boldsymbol{\mu}$, after observing a wage x of the currently available job offer as a sample, information is improved as $\boldsymbol{\mu}(x)$ and the decision maker decides whether to take this job offer. If she does not accept this job offer, the time period moves forward by one unit and this chain will make a transition into a new state according to $\mathbf{P}$, and information about the unobservable state becomes $\overline{\boldsymbol{\mu}(x)}$.

When there are n stages to go, suppose that prior information about the unobservable state is $\boldsymbol{\mu}$ and a wage of the current job offer is not observed at this time. Let $\tilde{P}(\boldsymbol{\mu},n,m)_j$ be a probability to make a transition into State j after m additional transitions by employing the optimal policy ($i,j=1,2,\dots,K$ and $m \le n$, $n,m=1,2,\dots$), i.e., the decision-maker observed the first $n-m$ job offers without accepting each of them under the optimal policy and the state of the chain becomes j after $n-m$ transitions. Since the optimal policy varies according to the remaining number of job offers, these $\tilde{P}(\boldsymbol{\mu},n,m)_j$ depend on n.

In order to consider these $\tilde{P}(\boldsymbol{\mu},n,m)_j$, we initially observe the case $m=1$. Suppose that a wage x of the currently available job offer is observed as a sample and this job is not accepted by the decision-maker. In this time, we put $\boldsymbol{\mu}^* = \boldsymbol{\mu}(x)$. Let $\tilde{P}'(\boldsymbol{\mu}^*,n,1)_j$ be a probability to be in State j after making the next transition ($j=1,2,\dots,K$), then $\tilde{\mathbf{P}}'(\boldsymbol{\mu}^*,n,1) = (\tilde{P}'(\boldsymbol{\mu}^*,n,1)_1,\dots,\tilde{P}'(\boldsymbol{\mu}^*,n,1)_K)$ satisfies the equations that $\tilde{P}'(\boldsymbol{\mu}^*,n,1)_j = \sum_{i=1}^K \mu_i^* p_{i,j}$ and $\tilde{\mathbf{P}}'(\boldsymbol{\mu}^*,n,1) = \boldsymbol{\mu}\mathbf{P} = \overline{\boldsymbol{\mu}^*}$ since the time period moves forward by one unit and this chain will make a transition into a new state whenever $x \in C(\boldsymbol{\mu},n)$. Since $\boldsymbol{\mu}^* = \boldsymbol{\mu}(x)$, we have $\tilde{P}(\boldsymbol{\mu},n,1)_j = \int_{C(\boldsymbol{\mu},n)} \tilde{P}'(\boldsymbol{\mu}(x),n,1)_j dF_{\boldsymbol{\mu}}(x) = \int_{C(\boldsymbol{\mu},n)} \sum_{i=1}^K \mu(x)_i p_{i,j}\, dF_{\boldsymbol{\mu}}(x)$. We, therefore, have the relationship between $\tilde{\mathbf{P}}(\boldsymbol{\mu},n,1) = (\tilde{P}(\boldsymbol{\mu},n,1)_1,\dots,\tilde{P}(\boldsymbol{\mu},n,1)_K)$ and $\tilde{\mathbf{P}}'(\boldsymbol{\mu},n,1)$ as

$$\tilde{\mathbf{P}}(\boldsymbol{\mu},n,1) = \int_{C(\boldsymbol{\mu},n)} \tilde{\mathbf{P}}'(\boldsymbol{\mu}(x),n,1)\, dF_{\boldsymbol{\mu}}(x) = \int_{C(\boldsymbol{\mu},n)} \overline{\boldsymbol{\mu}(x)}\, dF_{\boldsymbol{\mu}}(x).$$

Next we consider the general case where $m>1$. When prior information is $\boldsymbol{\mu}$ and n job offers remain, $\tilde{P}(\boldsymbol{\mu},n,m)_j$ is a probability to make a transition into State j after m additional transitions under the optimal policy ($i,j=1,2,\dots,K$ and $m \le n$, $n,m=1,2,\dots$). In this case, a new job offer will appear with a wage depending on the state of the chain, and then make a decision whether to accept this job offer. Since this job search problem continues whenever $x \in C(\boldsymbol{\mu},n)$, it is easy to show that $\tilde{P}(\boldsymbol{\mu},n,m)_j$ satisfies the recursive equation as

$$\tilde{P}(\boldsymbol{\mu},n,m)_j = \int_{C(\boldsymbol{\mu},n)} \tilde{P}\left(\overline{\boldsymbol{\mu}(x)},n-1,m-1\right)_j dF_{\boldsymbol{\mu}}(x) \tag{9}$$

similarly to the case where $m=1$, and $\tilde{\mathbf{P}}(\boldsymbol{\mu},n,m) = (\tilde{P}(\boldsymbol{\mu},n,m)_1,\dots,\tilde{P}(\boldsymbol{\mu},n,m)_K)$. Because $\int_{S(\boldsymbol{\mu},n)} dF_{\boldsymbol{\mu}}(x)$ is a probability to accept the current job offer at this stage, we have $\sum_{j=1}^K \tilde{P}(\boldsymbol{\mu},n,m)_j \le 1$. Furthermore, $\boldsymbol{\mu} \succeq \boldsymbol{\nu}$ implies $C(\boldsymbol{\mu},n) \subset C(\boldsymbol{\nu},n)$ by Lemma 6, i.e., the probability to continue at some stage decreases as μ increases, and, on the other hand, the probability to make a transition into the lower class increases as μ increases. When the state of the chain is observable to the decision maker, Proposition 1 yields that $\bar{\mathbf{P}}(n, m)$ is TP_2. On the contrary to this case, it is difficult to show such a property for $\tilde{\mathbf{P}}(\mu, n, m)$, since the probabilities $\tilde{P}(\overline{\mu(x)}, n-1, m-1)_j$ vary according to the observed wage x of an arriving new job offer. This comes from the fact that the current state is not directly observed by the decision-maker and she only knows information regarding what a present state is.

In this paper, we introduced some definitions (Definitions 1–4) by using the TP_2 and used several properties concerning these orders under Assumptions 1 and 2 since the properties of the job search problem on a partially observable Markov chain are considered. But here we skip the proofs due to limitations of space. Concerning the TP_2, Kijima [11] and Kijima and Ohnishi [12,13] also investigate the properties of TP_2 from the financial optimization.

4. CONCLUSION

In this paper, a job search problem on a partially observable Markov chain is considered, where the current state is not known directly. This problem can be considered to sell an asset of one company as watching the movement of the stock market. It is usual that the business situation of this company cannot be directly observable, and these conditions are estimated through the stock price of this company in the stock market. If there is a state designating the default, then this problem is what time the stocks are sold off before becoming bankrupt, and the probability to become bankrupt is also observed. When the current state is observable, this probability has TP_2 property, but it is not true when the current state is not known directly. The difficulty of these problems comes from the incompleteness about the knowledge of the current state as was shown in this paper. These problems can be considered when the state changes according to a partially observable Markov process or the diffusion process which is characterized by the total positivity. It is also possible to apply this model to a sequential decision problem to make decisions before entering some particular states.

REFERENCES

1. S.A. Lippman and J.J. McCall, Job search in a dynamic economy, *Journal of Economic Theory* **12**, 365–390, (1976).
2. S. Karlin and J.L. McGregor, Classical diffusion process and total positivity, *Journal of Mathematical Analysis and Applications* **1**, 163–183, (1960).
3. S. Karlin, *Total Positivity*, Stanford University Press, Stanford, CA, (1968).
4. S. Karlin and Y. Rinott, Total positivity properties of absolute value multinomial variables with applications to confidence interval estimates and related probabilistic inequalities, *The Annals of Statistics* **9**, 1035–1049, (1981).
5. T. Nakai, A generalization of multivariate total positivity of order two with an application to Bayesian learning procedure, *Journal of Information & Optimization Sciences* **23**, 163–176, (2002).
6. T. Nakai, A sequential stochastic assignment problem in a partially observable Markov chain, *Mathematics of Operations Research* **11**, 230–240, (1986).
7. T. Nakai, An optimal selection problem on a partially observable Markov chain, In *Stochastic Modelling in Innovative Manufacturing, Lecture Notes in Economics and Mathematical Systems, Volume 445*, (Edited by A.H. Christer, S. Osaki and L.C. Thomas), pp. 140–154, Springer-Verlag, Berlin, (1996).
8. T. Nakai, An optimal assignment problem for multiple objects per period—Case of a partially observable Markov chain, *Bulletin of Informatics and Cybernetics* **31**, 23–34, (1999).
9. T. Nakai, A generalized TP_2 and a sequential stochastic model on a partially observable Markov process, In *Probabilistic Methods in Discrete Mathematics—Proceedings of the Fifth International Petrozavodsk Conference*, (Edited by V.F. Kolchin, V.Ya. Kozlov, V.V. Mazalov, Yu.L. Pavlov and Yu.V. Prokhorov), pp. 291–302, VSP Publishers, The Netherlands, (2002).
10. F. De Vylder, Duality theorem for bounds in integrals with applications to stop loss premiums, *Scandinavian Actuarial Journal*, 129–147, (1983).
11. M. Kijima, Monotonicities in a Markov chain model for valuing corporate bonds subject to credit risk, *Mathematical Finance* **8**, 229–247, (1998).
12. M. Kijima and M. Ohnishi, Portfolio selection problems via the bivariate characterization of stochastic dominance relations, *Mathematical Finance* **6**, 237–277, (1996).

13. M. Kijima and M. Ohnishi, Stochastic orders and their applications in financial optimization, *Mathematical Methods of Operations Research* **50**, 351–372, (1999).